Appendix

A Derivation

According to the Equation (1), the q_1 and q_2 are needed to calculate the weighting function of caustics. This appendix describes the derivation of the q_1 and q_2 in view of Russian roulette and oversampling.

At first, we show derivation of the q_1. The directions of nearest photons at the position x are approximation of sample directions according to $L_c(x, \vec{\omega}) \cos\theta$. The $L_c(x, \vec{\omega})$ is the incident radiance of caustics at the position x and the direction $\vec{\omega}$. The θ is the angle between the $\vec{\omega}$ and the surface normal. Therefore, the ideal PDF of nearest photons at x is represented by the following p_1.

$$p_1 = \frac{L_c(x, \vec{\omega}) \cos\theta}{\int_\Omega L_c(x, \vec{\omega}) \cos\theta d\vec{\omega}} \tag{A.1}$$

The $L_c(x, \vec{\omega})$ can be represented by the following equation

$$L_c(x, \vec{\omega}) = L_e(x', \vec{\omega}')\gamma \tag{A.2}$$

where $L_e(x', \vec{\omega}')$ is the emitted radiance of the light source at the position x' and the direction $\vec{\omega}'$, $L_e(x', \vec{\omega}') = \infty$ when a photon is emitted from a point light source or a parallel light source, and γ is the transmittance of the path from the light source x' to the current position x. And it is well known that the denominator of Equation (A.1) is estimated by [Jensen 2001].

$$\int_\Omega L_c(x, \vec{\omega}) \cos\theta d\vec{\omega} \approx \sum_j^{m_1} \frac{\Delta\Phi_j}{\Delta A} \tag{A.3}$$

m_1 is the number of nearest photons, $\Delta\Phi_j$ is the power of photon j, and ΔA is the area covered by the nearest photons. For simplification, we assume that the all photons have the same power. Then we obtain the following equation

$$\sum_j^{m_1} \frac{\Delta\Phi_j}{\Delta A} = \frac{m_1 \Phi}{N \Delta A} \tag{A.4}$$

where Φ is the total radiant flux of the light sources, and N is the number of emitted photons. Therefore, the estimated PDF $\hat{p}_1$ is represented by the following equation.

$$\hat{p}_1 = \frac{N L_e(x', \vec{\omega}')\gamma \cos\theta \Delta A}{m_1 \Phi} \tag{A.5}$$

The q_1 is given by product of the number of samples m_1 and the estimated PDF $\hat{p}_1$.

$$q_1 = m_1 \hat{p}_1 = \frac{N L_e(x', \vec{\omega}')\gamma \cos\theta \Delta A}{\Phi} \tag{A.6}$$

Next, we consider Russian roulette in sample rays (final gather rays) according to the BRDF. When a sample ray hits a perfect specular surface, we can use Russian roulette to decide if the ray should be traced recursively or not. And it is efficient that the transmittance γ is used to the probability of Russian roulette. In this case, the q_2 is given by

$$q_2 = m_2 p_2 \gamma \tag{A.7}$$

where m_2 is the number of sample rays, and p_2 is the PDF of the BRDF model at x. Then, the γ is found in the both Equation (A.6) and Equation (A.7). By the Equation (1), the γ is eliminated in calculating the weighting function. Eventually, we can use the following q_1 and q_2.

$$q_1 = \frac{N L_e(x', \vec{\omega}') \cos\theta \Delta A}{\Phi} \tag{A.8}$$

$$q_2 = m_2 p_2 \tag{A.9}$$

At last, we consider the case that oversampling is used. If oversampling is used, the total number of sample rays per pixel increase with the number of oversamples. On the other hand, the total number of nearest photons per pixel is approximately same with m_1 regardless of the number of oversamples, because most of nearest photons are the same for every sampling point in one pixel when the ΔA is sufficiently larger than the pixel size. Therefore, we extend the Equation (A.8).

$$q_1 = \frac{N L_e(x', \vec{\omega}') \cos\theta \Delta A}{\Phi \alpha} \tag{A.10}$$

The α is a parameter for oversampling. When a ray first intersects a non-perfect specular surface, α is the same as number of oversamples per pixel, otherwise $\alpha = 1$. By introducing the α, we obtain the appropriate weighting function regardless of number of oversamples.