

# Supplementary Document: Real-time Rendering of Layered Materials with Anisotropic Normal Distributions

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## A ADDITIONAL FORMULAS

### A.1 Derivation of Jacobian Matrices

To derive Jacobian matrices, we partly followed the derivation by Stam [2001]. Different from his derivation, we derived an approximate solution for the Jacobian matrices over the region near to the origin of  $\mathcal{P}$ , while Stam derived the exact solution only at the origin. Without loss of generality, we can assume incident direction  $\omega_i$  as  $(\theta_i, 0)$ . Let  $\omega_r$  and  $\omega_t$  be directions for reflection and refraction, respectively. We denote the directions  $\omega_i$ ,  $\omega_r$ ,  $\omega_t$ , and  $\mathbf{h}$  as follows:

$$\begin{aligned}\omega_i &= (\sin \theta_i, 0, \cos \theta_i), \\ \omega_r &= (x_r, y_r, z_r), \\ \omega_t &= (x_t, y_t, z_t), \\ \mathbf{h} &= (x_h, y_h, z_h).\end{aligned}$$

Let  $\eta$  be a relative refractive index between two interfaces, we can write  $\omega_r$  and  $\omega_t$  as follows:

$$\begin{aligned}\omega_r &= 2(\omega_i \cdot \mathbf{h})\mathbf{h} - \omega_i, \\ \eta\omega_t &= \left( \omega_i \cdot \mathbf{h} - \sqrt{(\omega_i \cdot \mathbf{h})^2 + \eta^2 - 1} \right) \mathbf{h} - \omega_i.\end{aligned}$$

Using these equations, we can obtain the projected 2D coordinates  $(x_r, y_r)$  and  $(x_t, y_t)$  of  $\omega_r$  and  $\omega_t$ :

$$\begin{cases} x_r = 2Ax_h - \sin \theta_i \\ y_r = 2Ay_h \\ \eta x_t = \left( A - \sqrt{A^2 + \eta^2 - 1} \right) x_h - \sin \theta_i \\ \eta y_t = \left( A - \sqrt{A^2 + \eta^2 - 1} \right) y_h \end{cases}$$

where  $A = x_h \sin \theta_i + \cos \theta_i \sqrt{1 - x_h^2 - y_h^2}$ .

Therefore, for reflection, the Jacobian matrix is obtained as in the main body of the paper. For refraction, the Jacobian matrix is calculated as follows:

$$\mathbf{J}_t = \begin{bmatrix} \frac{\partial x_t}{\partial x_h} & \frac{\partial x_t}{\partial y_h} \\ \frac{\partial y_t}{\partial x_h} & \frac{\partial y_t}{\partial y_h} \end{bmatrix},$$

$$\begin{aligned}\eta \frac{\partial x_t}{\partial x_h} &= A - \sqrt{A^2 + \eta^2 - 1} + x_h \frac{\partial A}{\partial x_h} \left( 1 - \frac{2A}{\sqrt{A^2 + \eta^2 - 1}} \right) \\ \eta \frac{\partial x_t}{\partial y_h} &= x_h \frac{\partial A}{\partial y_h} \left( 1 - \frac{2A}{\sqrt{A^2 + \eta^2 - 1}} \right) \\ \eta \frac{\partial y_t}{\partial x_h} &= y_h \frac{\partial A}{\partial x_h} \left( 1 - \frac{2A}{\sqrt{A^2 + \eta^2 - 1}} \right) \\ \eta \frac{\partial y_t}{\partial y_h} &= A - \sqrt{A^2 + \eta^2 - 1} + y_h \frac{\partial A}{\partial y_h} \left( 1 - \frac{2A}{\sqrt{A^2 + \eta^2 - 1}} \right)\end{aligned}$$

where  $\begin{cases} \frac{\partial A}{\partial x_h} = \sin \theta_i - \frac{x_h \cos \theta_i}{\sqrt{1 - x_h^2 - y_h^2}}, \\ \frac{\partial A}{\partial y_h} = -\frac{y_h \cos \theta_i}{\sqrt{1 - x_h^2 - y_h^2}} \end{cases}$

As we wrote in the main body of the paper, we assume  $x_h$ ,  $y_h$ , and  $\theta_i$  are small enough that we can ignore the second- and higher-order terms of  $x_h$ ,  $y_h$ , and  $\sin \theta_i$ . Then, we can approximate  $\mathbf{J}_t$  as follows:

$$\begin{aligned}\mathbf{J}_t &\approx \frac{1}{\eta} \begin{bmatrix} \cos \theta_i - \sqrt{\cos^2 \theta_i + \eta^2 - 1} & 0 \\ 0 & \cos \theta_i - \sqrt{\cos^2 \theta_i + \eta^2 - 1} \end{bmatrix} \\ &= \frac{1}{\eta} \begin{bmatrix} \cos \theta_i - \cos \theta_t & 0 \\ 0 & \cos \theta_i - \cos \theta_t \end{bmatrix}\end{aligned}$$

Thus, the Jacobian matrix for refraction is also diagonal and its diagonal entries are the same.

### A.2 Adding-Doubling for Two-layer Materials

For two-layer materials, Belcour [2018] provided the result of the adding-doubling method in Section 5 of his paper. To extend their formulas using our result for anisotropic distribution is easy. By replacing the scalar variances  $\sigma_{ij}^{\{T,R\}}$  with covariance matrices  $\Sigma_{ij}^{\{T,R\}}$ . The series of interactions that are possible in two-layer materials are only  $R$  and  $TR^+T$ . The atomic operators for  $R$  are given by

$$\begin{aligned}e^R &= r_{12}, \\ \mu^R &= -\mu_i, \\ \Sigma^R &= r_{12} \Sigma_{12}^R.\end{aligned}$$

For  $TR^+T$ , the atomic operators are obtained as follows:

$$e^{TR^+T} = \frac{t_{12}r_{23}t_{12}}{1 - r_{23}r_{12}},$$

$$\mu^{TR^+T} = -\mu_i,$$

$$\Sigma^{TR^+T} = \frac{t_{12}r_{23}t_{12}}{1 - r_{23}r_{12}} \left[ \Sigma_{12}^T + \Sigma_{21}^T + K_{21} \left( \Sigma_{23}^R + \frac{r_{23}r_{21}}{1 - r_{23}r_{21}} \Sigma_{21}^R \right) \right].$$

In these formulas,  $r_{jk}$  and  $t_{jk}$  denote reflection and transmission coefficients between  $j$ -th and  $k$ -th interfaces, and  $K_{jk}$  is a transmission scaling factor which scales the roughness parameters. As explained in the main body of the paper,  $\Sigma_{12}^{\{R,T\}}$  can be obtained as follows:

$$\Sigma_{12}^{\{R,T\}} = \begin{bmatrix} \mathbf{t}_x & \mathbf{t}_y \end{bmatrix}^\top \begin{bmatrix} \sigma_{12,x}^{\{R,T\}} & 0 \\ 0 & \sigma_{12,y}^{\{R,T\}} \end{bmatrix} \begin{bmatrix} \mathbf{t}_x & \mathbf{t}_y \end{bmatrix},$$

$$\sigma_{12,\{x,y\}}^R = h\left(\alpha_{\{x,y\}}\right), \quad \sigma_{12,\{x,y\}}^T = h\left(s \times \alpha_{\{x,y\}}\right).$$

## REFERENCES

- L. Belcour. 2018. Efficient rendering of layered materials using an atomic decomposition with statistical operators. *ACM Trans. Graph.* 37, 4, Article 73 (2018), 15 pages. <https://doi.org/10.1145/3197517.3201289>
- J. Stam. 2001. An illumination model for a skin layer bounded by rough surfaces. In *Eurographics Workshop on Rendering*. 39–52. <https://doi.org/10.2312/EGWR/EGWR01/039-052>

(Appendix B starts from the next page)

## B ADDITIONAL RESULTS

### B.1 Results for varying roughness parameters

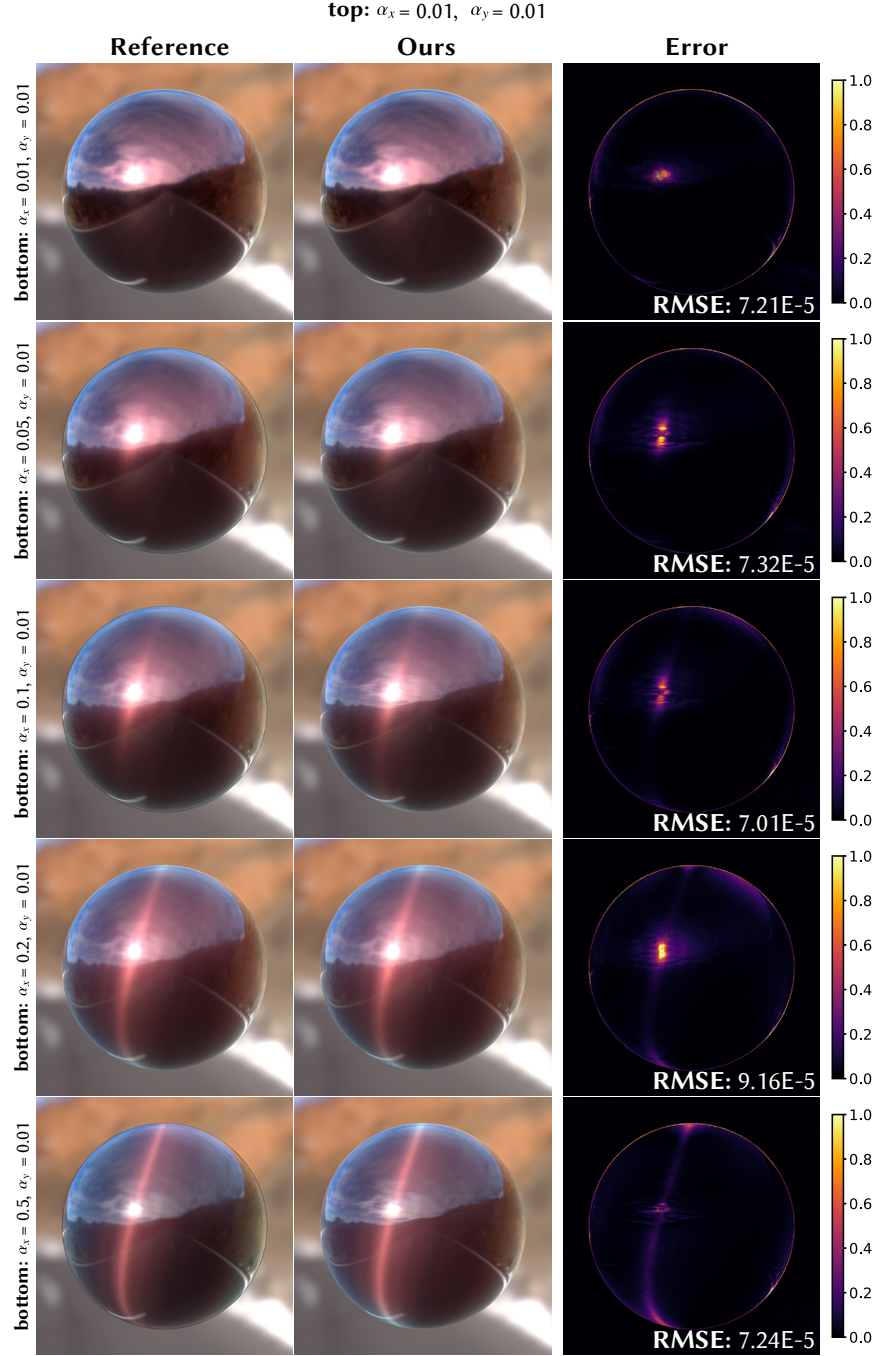


Figure 1: Rendering results with varying roughness parameters on the bottom layer ranging from 0.01 to 0.5. The roughness parameters of the top layer are fixed at  $(\alpha_x, \alpha_y) = (0.01, 0.01)$

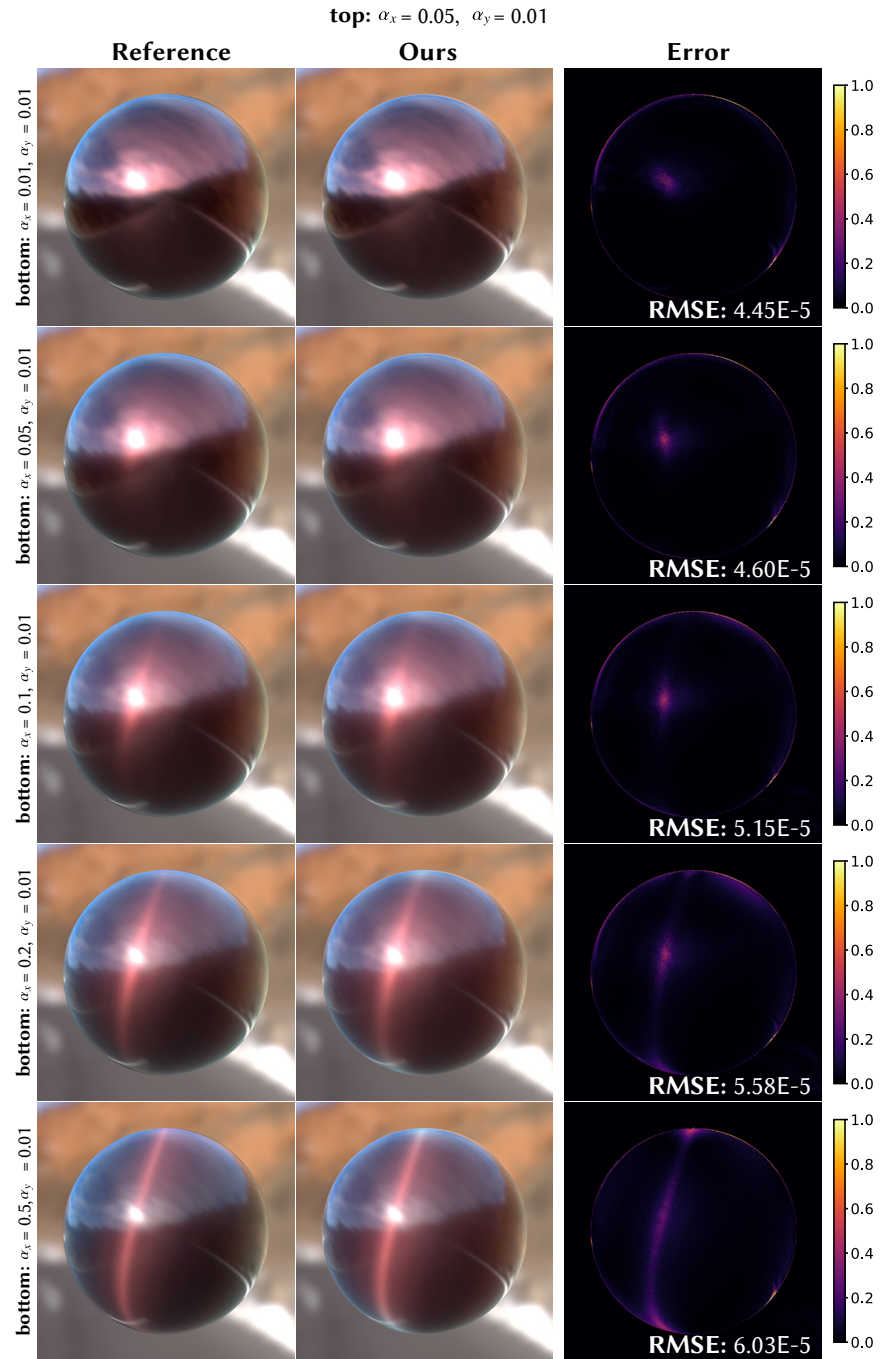
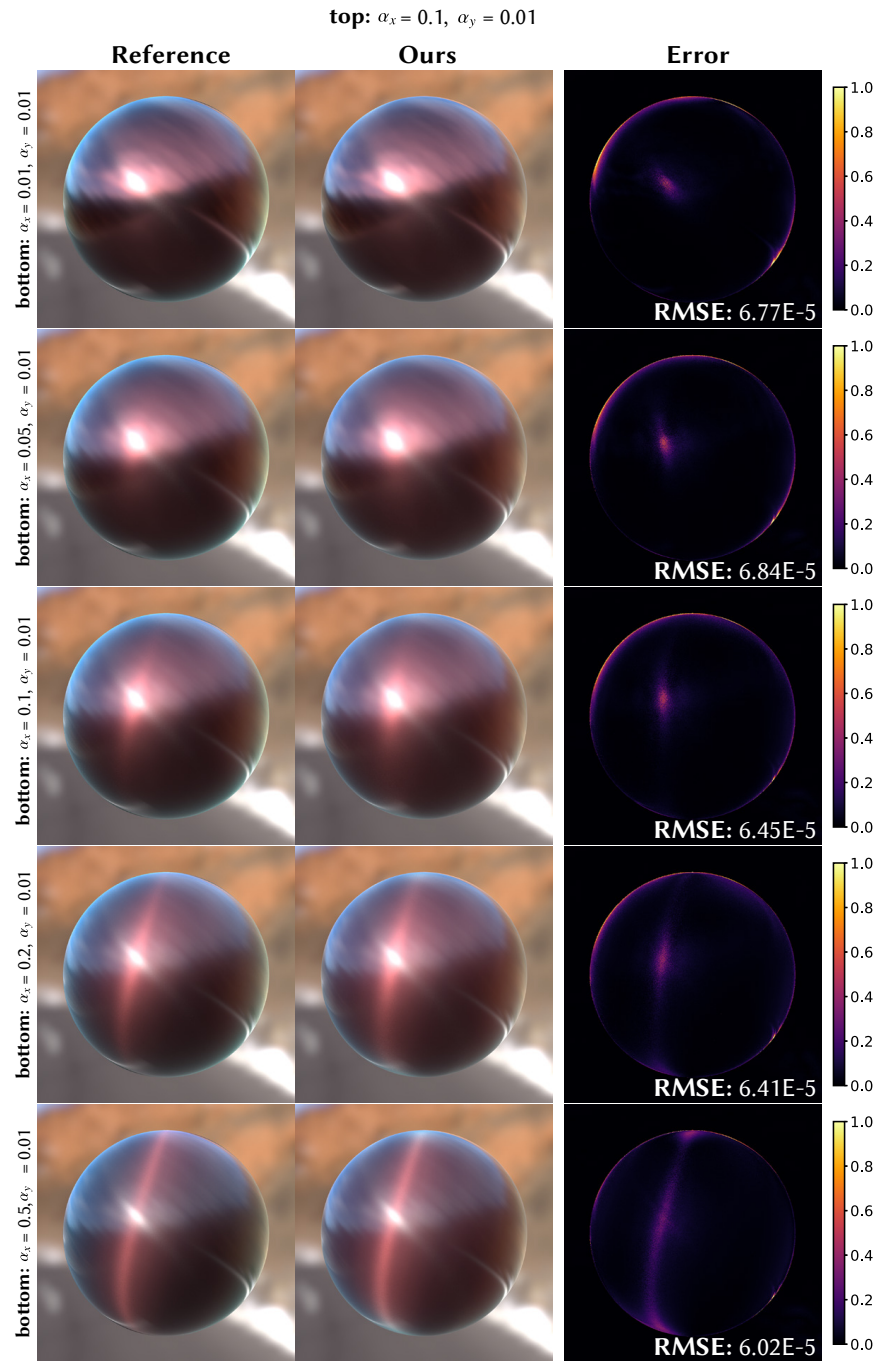


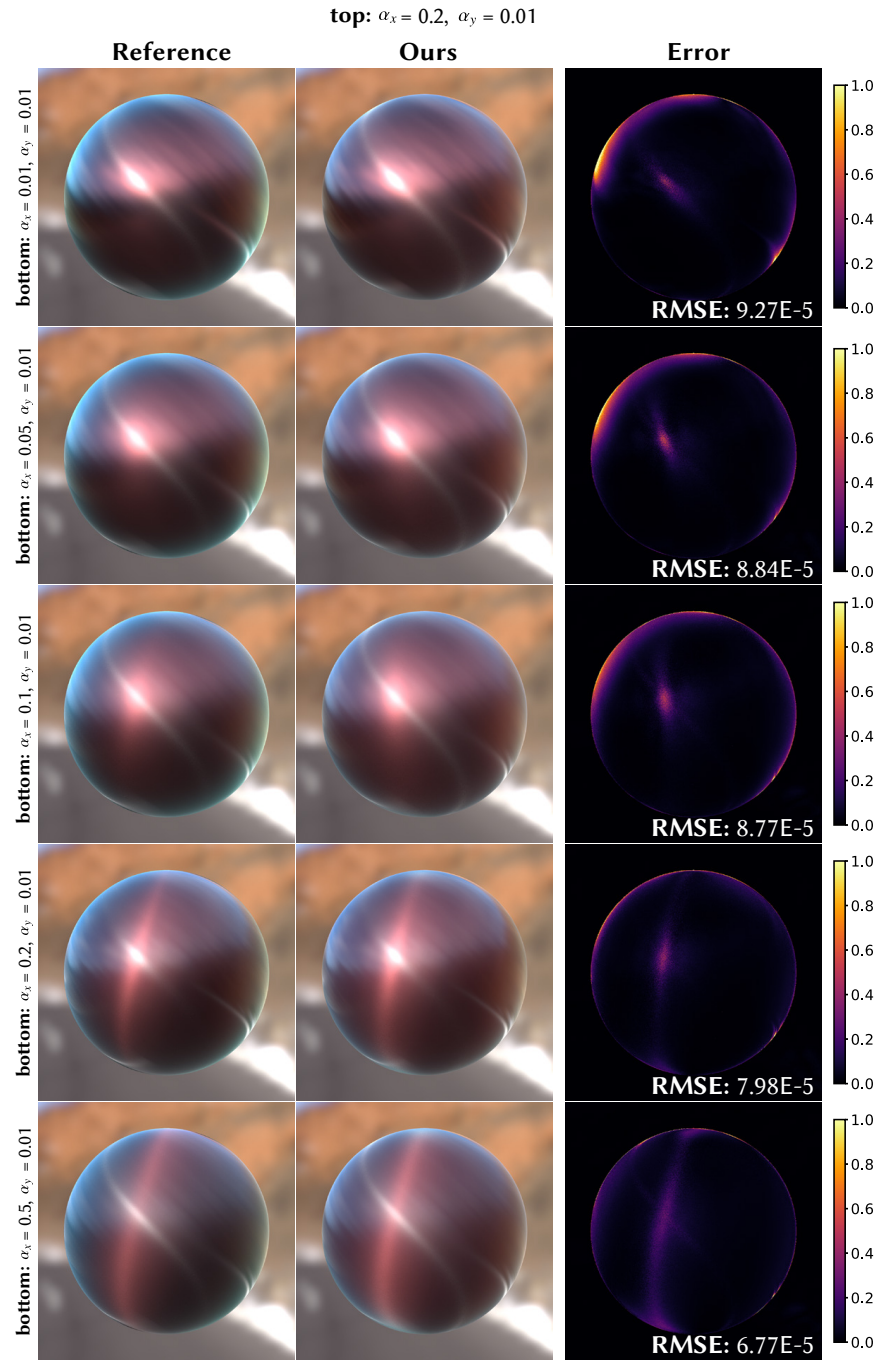
Figure 2: Rendering results with varying roughness parameters on the bottom layer ranging from 0.01 to 0.5. The roughness parameters of the top layer are fixed at  $(\alpha_x, \alpha_y) = (0.05, 0.01)$

## B.2 Results for varying rotation of local coordinate system

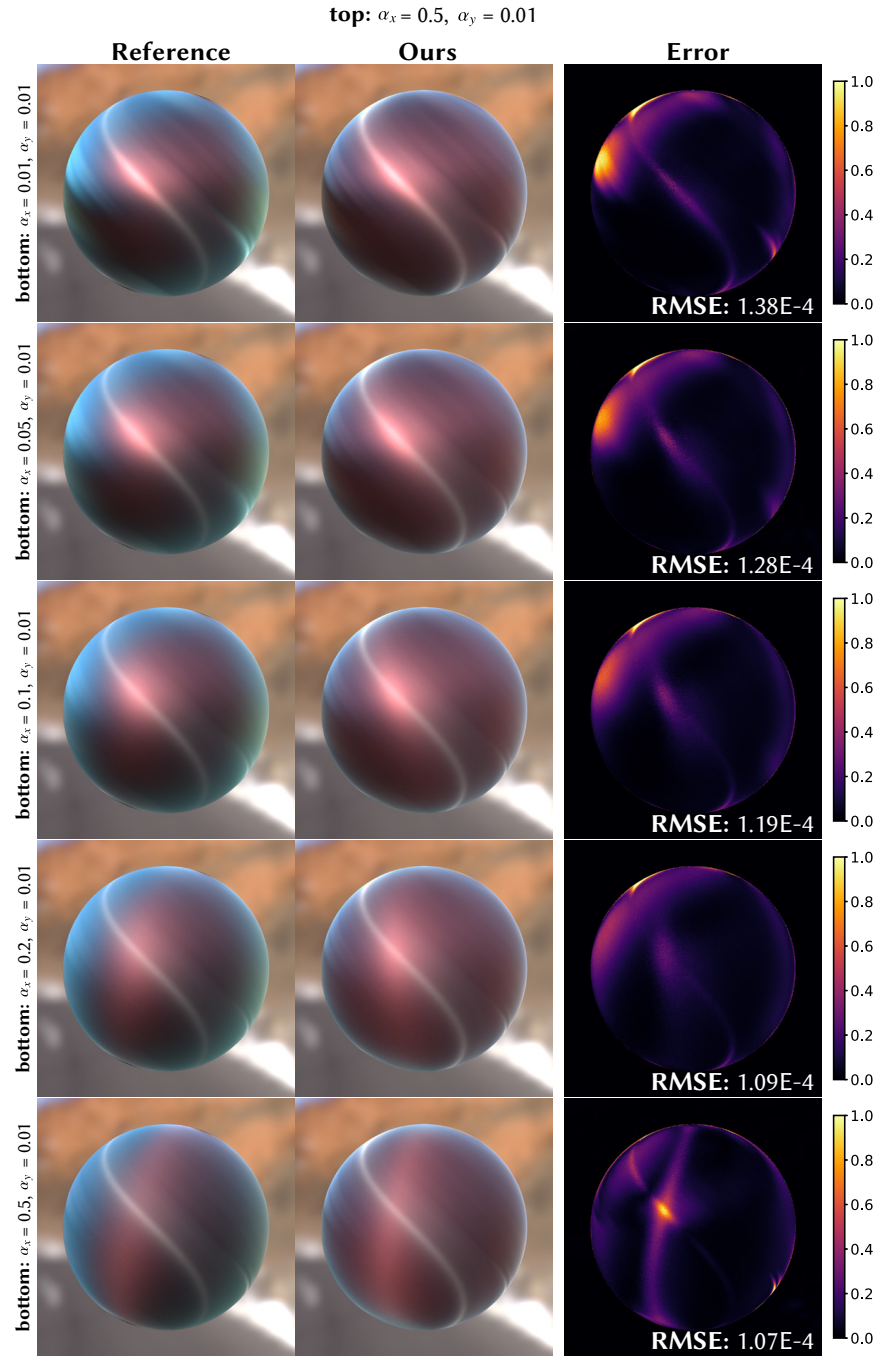


**Figure 3: Rendering results with varying roughness parameters on the bottom layer ranging from 0.01 to 0.5. The roughness parameters of the top layer are fixed at  $(\alpha_x, \alpha_y) = (0.1, 0.01)$**





**Figure 4: Rendering results with varying roughness parameters on the bottom layer ranging from 0.01 to 0.5. The roughness parameters of the top layer are fixed at  $(\alpha_x, \alpha_y) = (0.2, 0.01)$**



**Figure 5: Rendering results with varying roughness parameters on the bottom layer ranging from 0.01 to 0.5. The roughness parameters of the top layer are fixed at  $(\alpha_x, \alpha_y) = (0.5, 0.01)$**

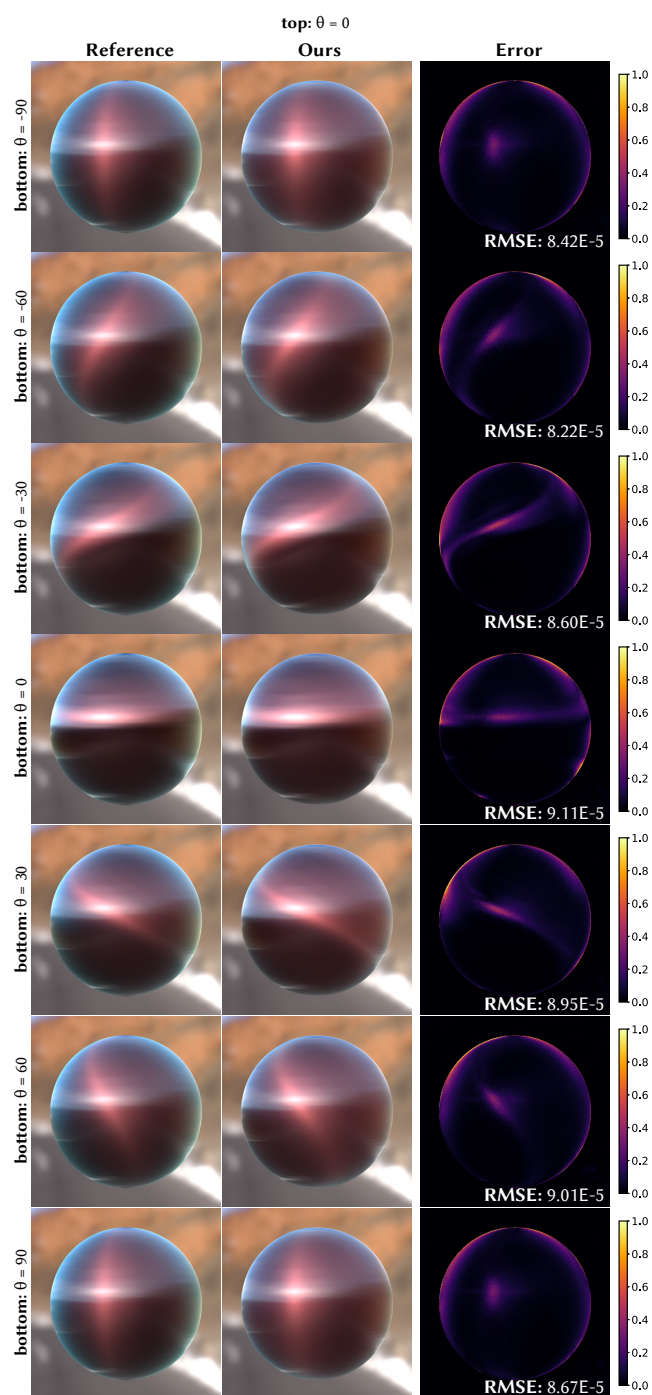


Figure 6: Rendering results for rotated local coordinate systems for the *bottom* layer, while the local coordinate system of the top layer is fixed.



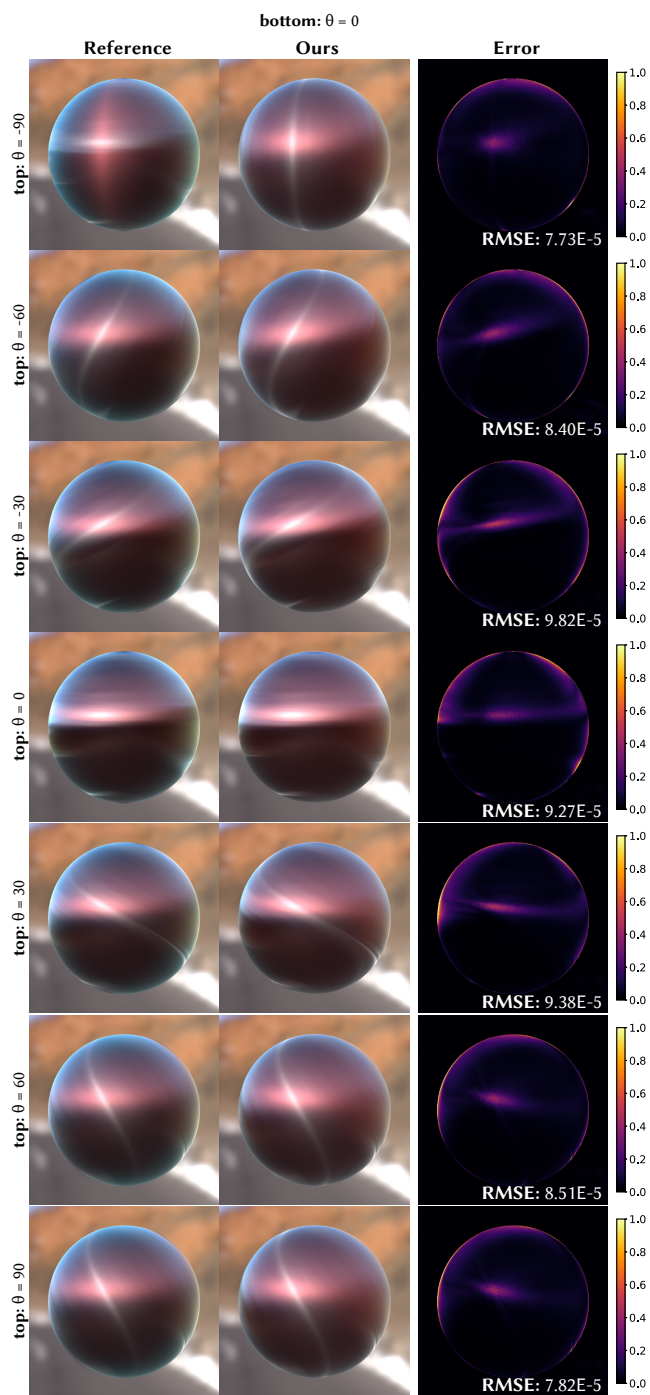


Figure 7: Rendering results for rotated local coordinate systems for the *top* layer, while the local coordinate system of the bottom layer is fixed.